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Exact solution of the displacement boundary-value problem of elasticity for a torus

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Abstract. This paper presents an exact analytical solution to the displacement boundary-value problem of elasticity for a torus. The introduced form of the general solution of elastostatics equations allows to solve exactly a broad class of boundary-value problems in coordinate systems with incomplete separation of variables in the harmonic equation. The original boundary-value problem for a torus is reduced to infinite systems of linear algebraic equations with tridiagonal matrices. An analytical technique for solving systems of diagonal form is developed. Uniqueness of the solutions of vector boundary-value problems involving the generalized Cauchy-Riemann equations is investigated, and it is shown that the obtained solution for the displacement boundary-value problem for a torus is unique due to the specific properties of the suggested general solution. The analogy between problems of elastostatics and steady Stokes flows is demonstrated, and the developed elastic solution is used to solve the Stokes problem for a torus.

Key words: exact solution, multiply-connected regions, Stokes flows, toroidal coordinates, torus

1. Introduction

This paper investigates solutions of vector boundary-value problems of elasticity for bodies of a complex geometry, which can be described by toroidal and bispherical coordinates. Due to the complexity of the body of mathematics accompanying these coordinate systems, we restrict ourselves to considering only the displacement boundary-value problem of elasticity for a torus, which attracted our attention by the challenges related to the multiply-connectedness of toroidal domains. However, the general approach presented in this paper is equally applicable to all bodies described by the mentioned coordinate systems (*i.e.*, torus, lens-shaped body, spindle-shaped body, and bi-spheres).

Construction of the analytical solutions to the basic elasticity problems for spatial bodies is a challenging mathematical task due to the essentially vector nature of these problems. The most powerful technique yet is the application of curvilinear orthogonal coordinates along with the Fourier method of separation of variables. Over the past century there has been significant progress in this area, so now exact solutions are available for virtually all bodies described by separable coordinates. However, there exists a class of coordinate systems that do not admit a complete separation of variables in the harmonic equation, but still allow for solving scalar and vector boundary-value problems of mathematical physics. These curvilinear coordinate systems belong to the family of cyclidal coordinates [5, pp. 518–523], and the simplest of them are the toroidal and bispherical coordinates that can be used to describe a torus with a lens-shaped body and bi-spheres with a spindle-shaped body, respectively. The incomplete separation of variables in the Laplace equation in the cyclide coordinates resulted in a relatively rare application of these coordinate systems for solving applied and engineering problems. Mainly, the bispherical coordinates were used for solving problems for bi-spheres, while the toroidal coordinates were employed most often in boundary-value problems for a half-space, where two circular lines were separating boundary conditions of different type.

The first attempt to solve the basic elasticity problems for a torus goes back to Wangerin [2]. He suggested a general approach for solving elasticity problems for bodies of revolution, and applied it to the problems for a bi-axial ellipsoid, bi-spheres and a torus. The elasticity problems for tori and bi-spheres were reduced to infinite systems of linear algebraic equations of diagonal form with more than thirty diagonals. Unfortunately, in the problem for a torus the representation for the harmonics in the toroidal regions the functions were chosen incorrectly. Far later, Soloviev [3] used the technique of generalized analytical functions for solving axisymmetric problems for elastic tori and an elastic space with a toroidal cavity. The boundary-value problems were reduced to infinite systems with eight and twelve diagonals, which were solved numerically. In the papers by Podil'chuk and Kirilyuk [4, 5] the basic elasticity problems for a torus were reduced to infinite systems with five and fourteen diagonals. Their solutions were obtained numerically by successive truncation.

It was the unwieldy form of the infinite systems obtained in these papers that did not allow for an analytical solution, so we saw our objective in developing a solution that would yield equations of the simplest form. In view of that, the most interesting results were obtained regarding the Stokes-flow problem for toroidal bodies.

Being a linearized form of the Navier-Stokes equations in the approximation of low Reynolds number, the Stokes equations describe very slow ('creeping') flows of viscous incompressible fluids [6]. The reader should not be confused with addressing problems of hydrodynamics while considering the elastic framework. We will demonstrate that a direct analogy exists between the Stokes-flow problems and the elastostatics problems for incompressible solids. In fact, it can be shown that a steady Stokes flow can be regarded as an elastic equilibrium of a solid with Poisson ratio equal to 0·5. The simplifications of the axisymmetric Stokes model allowed Payne and Pell [7] to construct an analytical solution of the Stokes problem for a torus that did not involve infinite algebraic systems. However, the approach of [7] cannot be expanded to the general elasticity case. Another interesting approach was suggested by Wakiya [8], who obtained a tridiagonal infinite system for the axisymmetric Stokes problem for a torus, but still solved it numerically.

Here we present an exact solution to the displacement boundary-value problem for a torus. The next section introduces a general solution of elastostatics equations, which is a generalization of the solution for Stokes flow suggested by Wakiya [8], and can be applied to a broad class of problems in cyclidal coordinates. This general solution enables one to reduce the original boundary-value problem for a torus to a set of infinite algebraic systems with no more than three diagonals. In the fourth section we present an analytical technique for solving the resulting tridiagonal algebraic systems, which can be potentially applied to systems with a greater number of diagonals. Section 5 discusses the issues pertaining to the uniqueness of the obtained solution with respect to the double-connectedness of toroidal regions. It is shown that the introduced form of general solution of the elastostatics equations yields a single-valued solution of the displacement boundary-value problem for a torus.

As an illustration of the general approach, we revisit the Stokes problem for a torus in Section 6. However, this does not mean at all that the presented technique is incapable of solving some new problems. Quite the contrary, we want to highlight the important aspects of our approach on a simple, yet physically rich example, and at the same time use the obtained general solution to present a complete and comprehensive solution to this classical problem.

2. General solution of the elastostatics equation for cyclidal coordinate systems

The displacement boundary-value problem of elasticity consists in finding the vector of elastic displacement $\mathbf{u}$, which in a homogeneous isotropic solid must satisfy the elastostatics (Lamé) equation

$$2\frac{1-\nu}{1-2\nu}\text{grad div }\mathbf{u} - \text{curl curl }\mathbf{u} = 0, \tag{1}$$

subject to the boundary conditions

$$\mathbf{u}|_S = \mathbf{U}, \tag{2}$$

where ν is the Poisson ratio, S is the surface of the solid and $\mathbf{U}$ is the displacement vector prescribed on S.

A key role in the presented approach is played by the general solution of the elastostatics equation (1) that we will develop below. Introducing the notations

$$\text{div }\mathbf{u} = -\frac{1-2\nu}{2-2\nu}\vartheta \quad \text{and} \quad \gamma = \frac{1}{4(1-\nu)}, \tag{3}$$

we reduce Equation (1) to a non-homogeneous harmonic equation for the displacement vector $\mathbf{u}$:

$$\Delta\mathbf{u} = 2\gamma \text{ grad } \vartheta. \tag{4}$$

Observing that ϑ has to be a harmonic function, we write the general solution of (4) as the sum of a partial solution of the non-homogeneous equation (4) and the general solution for the corresponding homogeneous equation:

$$\mathbf{u} = \mathbf{B} + \gamma\,\mathbf{r}\vartheta, \quad \text{where} \quad \Delta\mathbf{B} = 0, \quad \Delta\vartheta = 0. \tag{5}$$

Functions $\mathbf{B}$ and ϑ are not independent; they have to satisfy the differential constraint

$$(\gamma+1)\vartheta + \gamma\,\mathbf{r}\cdot\text{ grad }\vartheta + \text{ div }\mathbf{B} = 0. \tag{6}$$

The choice of the general solution to be used in a particular problem is usually determined by the geometry of the problem. For example, the Papkovich-Neuber general solution

$$\mathbf{u} = \mathbf{B} - \frac{1}{4(1-\nu)}\text{grad}(\mathbf{r}\cdot\mathbf{B} + B_0), \quad \Delta\mathbf{B} = 0, \quad \Delta B_0 = 0,$$

works well in the elasticity problems for bodies whose surface represents a plane (*e.g.*, half-spaces or layers), but produces unwieldy equations for bodies with a complex geometry, such as a torus. In the following sections we will show that the derived general solution (5)–(6) leads to the equations of the simplest form in the boundary-value problem (1)–(2) for a torus. In general, given the similarities between the cyclidal coordinate systems, solution (5)–(6) will yield the simplest solutions for other bodies described by cyclidal coordinates, such as spindle- or lens-shaped bodies, or bi-spheres.

The solution of the elasticity problems for bodies of revolution can often be simplified if the original boundary-value problem is first formulated in cylindrical polar coordinates, and then solved using the special curvilinear coordinate system that describes the body surface. In accordance with this approach, we write functions $\mathbf{B}$, ϑ, and vector $\mathbf{u}$ as the Fourier series with respect to polar angle φ of the cylindrical coordinate system $\{r, z, \varphi\}$, whose z-axis coincides with the body's axis of revolution:

$$\mathbf{B} = \sum_{k=0}^{\infty}{}' \left\{ \mathbf{i}_r B_{rk}(r,z) \begin{matrix}\cos\\ \sin\end{matrix} k\varphi + \mathbf{i}_\varphi B_{\varphi k}(r,z) \begin{matrix}\sin\\ -\cos\end{matrix} k\varphi + \mathbf{i}_z B_{zk}(r,z) \begin{matrix}\cos\\ \sin\end{matrix} k\varphi \right\},$$

$$\vartheta = \sum_{k=0}^{\infty}{}' \vartheta_k(r,z) \begin{matrix}\cos\\ \sin\end{matrix} k\varphi,$$

$$\mathbf{u} = \sum_{k=0}^{\infty}{}' \left\{ \mathbf{i}_r u_k(r,z) \begin{matrix}\cos\\ \sin\end{matrix} k\varphi + \mathbf{i}_\varphi v_k(r,z) \begin{matrix}\sin\\ -\cos\end{matrix} k\varphi + \mathbf{i}_z w_k(r,z) \begin{matrix}\cos\\ \sin\end{matrix} k\varphi \right\}, \tag{7}$$

Here $\mathbf{i}_r$, $\mathbf{i}_\varphi$, and $\mathbf{i}_z$ are the basis vectors of the cylindrical coordinates, and the primed summation $\sum'$ indicates that the first term of the sum (in our case, the term with $k = 0$) is halved.

Harmonicity of the functions $\mathbf{B}$ and ϑ determines the equations to be satisfied by their Fourier components. Noting that $B_{\varphi 0} = 0$, for all $k \geq 0$, we have

$$\Delta_{k\pm 1}(B_{rk} \pm B_{\varphi k}) = 0, \quad \Delta_k B_{zk} = 0, \quad \Delta_k \vartheta_k = 0, \tag{8}$$

where the operator Δ_k is defined by

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{k^2}{r^2}.n$$

The Fourier components of $\mathbf{B}$ can be readily expressed as

$$B_{rk} = \phi_{k+1} + \psi_{k-1}, \quad B_{\varphi k} = \phi_{k+1} - \psi_{k-1}, \quad B_{zk} = \chi_k,$$

where $\psi_{-1} = 0$ and functions ϕ_{k+1}, ψ_{k-1}, and χ_k solve equations

$$\Delta_{k+1}\phi_{k+1} = 0, \quad \Delta_{k-1}\psi_{k-1} = 0, \quad \Delta_k \chi_k = 0. \tag{9}$$

By equality (5), the Fourier transforms (7) of the displacement vector $\mathbf{u}$ for harmonics with $k \geq 1$ can be written as

$$u_k = \phi_{k+1} + \psi_{k-1} + \gamma r \vartheta_k, \quad v_k = \phi_{k+1} - \psi_{k-1}, \quad w_k = \chi_k + \gamma z \vartheta_k, \tag{10a}$$

whereas in the axisymmetric case ($k = 0$) they are[1]

$$u_0 = \phi_1 + \gamma r \vartheta_0, \quad w_0 = \chi_0 + \gamma z \vartheta_0. \tag{10b}$$

In terms of the functions (9) the differential constraint (6) translates into

[1]We do not consider here the trivial realization of axisymmetric stress field in a body of revolution, known as *pure torsion*. In this case, the general solution contains only one non-zero component of the displacement vector $u_\varphi = -\frac{1}{2}v_0 = -\frac{1}{2}\phi_1$, with Equation (1) satisfied identically.

$$\gamma\left(1+\frac{1}{\gamma}+r\frac{\partial}{\partial r}+z\frac{\partial}{\partial z}\right)\vartheta_k+\left(\frac{\partial}{\partial r}+\frac{k+1}{r}\right)\phi_{k+1}+\left(\frac{\partial}{\partial r}-\frac{k-1}{r}\right)\psi_{k-1}+\frac{\partial\chi_k}{\partial z}=0. \tag{11}$$

Now determine the boundary conditions for the functions (9). Writing the Fourier expansion for vector **U** in the same way as

$$\mathbf{u}|_S=\sum_{k=0}^{\infty}{}'\left\{\mathbf{i}_r U_k(r,z)\begin{matrix}\cos\\ \sin\end{matrix}\,k\varphi+\mathbf{i}_\varphi V_k(r,z)\begin{matrix}\sin\\ -\cos\end{matrix}\,k\varphi+\mathbf{i}_z W_k(r,z)\begin{matrix}\cos\\ \sin\end{matrix}\,k\varphi\right\}, \tag{12}$$

then substituting the expressions (10) and (7) for **u** in (12), then after a minor rearrangement, we find that the functions (9) at the boundary of a solid must satisfy conditions

$$\gamma\vartheta_0\Big|_S=\left(-\frac{1}{r}\phi_1+\frac{1}{r}U_k\right)\Big|_S,\quad \chi_0\Big|_S=\left(\frac{z}{r}\phi_1-\frac{z}{r}U_k+W_k\right)\Big|_S, \tag{13a}$$

and for $k\geq 1$,

$$\begin{aligned}\gamma\vartheta_k|_S&=\left(-\frac{2}{r}\phi_{k+1}+\frac{1}{r}(U_k+V_k)\right)\Big|_S,\quad \psi_{k-1}|_S=(\phi_{k+1}-V_k)\Big|_S,\\ \chi_k|_S&=\left(2\frac{z}{r}\phi_{k+1}-\frac{z}{r}(U_k+V_k)+W_k\right)\Big|_S.\end{aligned} \tag{13b}$$

Thus, to find the Fourier components (10) of the displacement vector **u**, one has to solve the boundary-value problem (13) for the functions ϕ_{k+1}, ψ_{k-1}, χ_k, and ϑ_k, subject to the differential constraint (11). Note that (11) is by no means an equation to be solved: the general form of the functions it contains is already predetermined by Equations (8)–(9). However, the functions ϕ_{k+1}, ψ_{k-1}, χ_k, and ϑ_k are not independent; thus having them satisfy the constraint (11) eliminates the arbitrariness in the solution.

In what follows, we will apply toroidal coordinates to solve the boundary-value problem (13), where S is a toroidal surface.

3. Exact solution of the boundary-value problem in toroidal coordinates

The previous section demonstrated how the displacement boundary-value problem (1)–(2) for a body of revolution is reduced to the boundary-value problem (13), (11) for four functions (9), (8). The differential constraint (11) to be satisfied by these functions complements three equations of the boundary conditions (13), making a complete system for determining the functions ϕ_{k+1}, ψ_{k-1}, χ_k, and ϑ_k.[2]

A successful solution of a three-dimensional boundary-value problem can be achieved when an orthogonal and separable system of curvilinear coordinates exists, whose coordinate surfaces fit the surface of the boundary conditions. For problems involving toroidal bodies, such a system is given by toroidal coordinates[3], which belong to the family of the cyclidal coordinates [1, p. 666]. Being a conjugate system of revolution, the toroidal coordinates $\{\xi,\eta,\varphi\}$ relate to the cylindrical coordinates $\{r,z,\varphi\}$, where φ is the polar angle common for both systems, as

[2]In the axisymmetric case ($k=0$), there are three functions (ϕ_1, χ_0, and ϑ_0) and two boundary conditions (13a).
[3]The toroidal coordinates, as well as other cyclidal coordinate systems, are not completely separable; see below.

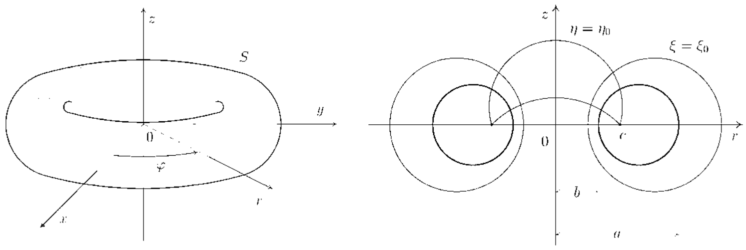

Figure 1. Torus and the toroidal coordiantes.

$$r = \frac{c \sinh \xi}{\cosh \xi - \cos \eta}, \quad z = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \quad 0 \le \xi < \infty, \ -\pi < \eta \le \pi, \tag{14}$$

where c is the metric parameter. The coordinate surfaces $\xi = \xi_0 =$ const represent toroids; orthogonal to them are surfaces $\eta = \eta_0 =$ const that generate spherical segments leaning on the circle $r = c$ in plane $z = 0$ (see Figure 1b). In the toroidal system, the origin has coordinates $\xi = 0, \eta = \pi$, the infinitely remote point of space is $\xi = 0, \eta = 0$, whereas points with $\xi = \infty$ correspond to the circle $r = c$, $z = 0$. Given the radii a and b of the outermost rim of the torus and the torus opening, the parameters c and ξ_0 can be found as

$$c = \sqrt{ab}, \quad \xi_0 = 2\text{arctanh}\,\sqrt{b/a}.$$

The distinguishing property of cyclidal coordinate systems, including the toroidal coordinates, in comparison with other separable coordinate systems (spherical, elliptical etc.), is the incomplete separation of variables in the Laplace equation and equations of type (9). A solution of the harmonic equation in a cyclidal system $\{\xi_1, \xi_2, \xi_3\}$ has the form $R(\xi_1, \xi_2, \xi_3)X_1(\xi_1)$ $X_2(\xi_2)X_3(\xi_3)$, where $R(\xi_1, \xi_2, \xi_3)$ is the *modulating factor* [1, pp. 518–519] that must be introduced to make possible the separation of eigenfunctions $X_i(\xi_i)$. The incomplete separation of variables in cyclidal coordinates can significantly complicate solution of even the basic problems of potential theory (see, for example, the paper by Lebedev [9] on potential-theory problems for a torus of elliptical cross-section).

In the toroidal and bi-spherical coordinate systems, the modulating factor has the form of the *Riemannian radical* $\sqrt{\cosh \xi - \cos \eta}$, and the eigenfunctions $X_i(\xi_i)$ are the trigonometric/hyperbolic functions and Legendre functions. The separation parameters for toroidal regions are chosen such that the general solutions of Equations (9) and (8) have the form

$$\begin{aligned}
\vartheta_k &= \frac{1}{c}\sqrt{\cosh \xi - \cos \eta}\ \sum_{n=0}^{\infty}{}' (A_{n,k} \cos n\eta + \tilde{A}_{n,k} \sin n\eta) L^k_{n-\frac{1}{2}}(\cosh \xi), \\
\phi_{k+1} &= \sqrt{\cosh \xi - \cos \eta}\ \sum_{n=0}^{\infty}{}' (B_{n,k} \cos n\eta + \tilde{B}_{n,k} \sin n\eta) L^{k+1}_{n-\frac{1}{2}}(\cosh \xi), \\
\psi_{k-1} &= \sqrt{\cosh \xi - \cos \eta}\ \sum_{n=0}^{\infty}{}' (D_{n,k} \cos n\eta + \tilde{D}_{n,k} \sin n\eta) L^{k-1}_{n-\frac{1}{2}}(\cosh \xi), \\
\chi_k &= \sqrt{\cosh \xi - \cos \eta}\ \sum_{n=0}^{\infty}{}' (C_{n,k} \sin n\eta - \tilde{C}_{n,k} \cos n\eta) L^k_{n-\frac{1}{2}}(\cosh \xi),
\end{aligned} \tag{15}$$

where $L^k_{n-\frac{1}{2}}(\cosh \xi)$ denote the Legendre functions of semi-integer index of the first or the second kind, $P^k_{n-\frac{1}{2}}(\cosh \xi)$ and $Q^k_{n-\frac{1}{2}}(\cosh \xi)$ (see, for instance, [10, pp. 433-443], [11, Chapter 3]. The following integral representations are available for the considered Legendre functions [11, pp. 155–160]:

$$P^k_{n-\frac{1}{2}}(\cosh\xi) = \frac{\Gamma(n+1/2+k)}{\pi\,\Gamma(n+1/2)} \int_0^{\pi} \frac{\cos k\tau \ \mathrm{d}\tau}{(\cosh\xi + \sinh\xi\ \cos\tau)^{1/2-n}}, \tag{16a}$$

$$Q^k_{n-\frac{1}{2}}(\cosh\xi) = \frac{(-1)^k}{2\sqrt{2\pi}}\,\Gamma(k+1/2) \int_{-\pi}^{\pi} \frac{\sinh^k\xi\ \cos n\tau\ \mathrm{d}\tau}{(\cosh\xi - \cos\tau)^{k+1/2}}. \tag{16b}$$

The asymptotic behavior of the Legendre functions with argument approaching infinity or unity [11, pp. 163–164] dictates that functions $Q^k_{n-\frac{1}{2}}(\cosh\xi)$ must be used in the expansions (15) for the inner problems, and $P^k_{n-\frac{1}{2}}(\cosh\xi)$ for outer problems for a torus.

The constants of integration $A_{n,k}, \ldots, D_{n,k}$, and $\tilde{A}_{n,k}, \ldots, \tilde{D}_{n,k}$ in the functions (15) have to be determined from the boundary conditions (12) and Equation (11). Let us first consider the equality (11). It is easy to verify that each summand in the differential constraint (11) satisfies the equation $\Delta_k(\cdot) = 0$, *i.e.*,

$$\Delta_k\left(\vartheta_k + \frac{\vartheta_k}{\gamma} + r\frac{\partial\vartheta_k}{\partial r} + z\frac{\partial\vartheta_k}{\partial z}\right) = 0, \ \ \ldots, \ \ \Delta_k\left(\frac{\partial\chi_k}{\partial z}\right) = 0,$$

and, consequently, the left-hand side of (11) is a solution to the equation $\Delta_k(\cdot) = 0$. Therefore, on inserting representations (15) in Equation (11), it transforms to

$$\sqrt{\cosh\xi - \cos\eta}\sum_{n=0}^{\infty}(\{\ldots\}\cos n\eta + \{\widetilde{\ldots}\}\sin n\eta)\, L^k_{n-\frac{1}{2}}(\cosh\xi) = 0, \tag{17}$$

where the terms in braces contain linear combinations of coefficients (15), tilded and untilded separately. Evidently, for Equation (17) to hold for all values of ξ and η, the terms in braces must be equal to zero. This yields two separate infinite sets of linear algebraic equations with respect to the tilded and untilded coefficients (15). To shorten the presentation, we consider here only the untilded constants:

$$\begin{aligned}
&\gamma\Big((1/2{+}1/\gamma)A_{0,k} + (k{+}1/2)A_{1,k}\Big) - (k{+}1/2)^2 B_{0,k}\\
&\quad -(k{+}1/2)(k{+}3/2)B_{1,k} - D_{1,k} + D_{0,k} + (k{+}1/2)C_{1,k} = 0,
\end{aligned}$$

$$\begin{aligned}
&\gamma\Big((n{+}k{+}1/2)A_{n+1,k} + (1{+}2/\gamma)A_{n,k} - (n{-}k{-}1/2)A_{n-1,k}\Big)\\
&\quad -(n{+}k{+}3/2)(n{+}k{+}1/2)B_{n+1,k} + 2(n{+}k{+}1/2)(n{-}k{-}1/2)B_{n,k}\\
&\quad -(n{-}k{-}1/2)(n{-}k{-}3/2)B_{n-1,k} - D_{n+1,k} + 2D_{n,k} - D_{n-1,k}\\
&\quad +(n{+}k{+}1/2)C_{n+1,k} - 2nC_{n,k} + (n{-}k{-}1/2)C_{n-1,k} = 0, \quad n \geq 1,
\end{aligned} \tag{18}$$

where $C^k_0 = 0$. The system for the tilded constants has the same form but does not contain the first equation for $n = 0$.

Now we make use of the boundary conditions (13). Note that equalities (13) represent the functions ϑ_k, ψ_{k-1}, and χ_k at the boundary $\xi = \xi_0$ in terms of the boundary values of function

ϕ_{k+1}. Consequently, substitution of the expansions (15) in the boundary conditions (13) yields an expression for the constants $A_{n,k}$, $C_{n,k}$, $D_{n,k}$ in terms of some new unknowns $x_{n,k}$, which are nothing but the normalized coefficients $B_{n,k}$ of function ϕ_{k+1}:[4]

$$\begin{aligned}
\gamma A_{0,k} &= -2\kappa_{0,k}\cosh\xi_0\; x_{0,k} + 2\kappa_{0,k}x_{1,k} + \kappa_{0,k}\, p_{0,k},\\
\gamma A_{n,k} &= -2\kappa_{n,k}\cosh\xi_0\; x_{n,k} + \kappa_{n,k}(x_{n+1,k} + x_{n-1,k}) + \kappa_{n,k}\, p_{n,k},\;\; n \geq 1,\\
B_{n,k} &= \lambda_{n,k}\, x_{n,k}, \quad n \geq 0; \quad C_{n,k} = \kappa_{n,k}(x_{n-1,k} - x_{n+1,k}) + \kappa_{n,k}\, s_{n,k}, \quad n \geq 1,\\
D_{n,k} &= \gamma_{n,k}\, x_{n,k} + \gamma_{n,k}\, q_{n,k}, \quad n \geq 0.
\end{aligned} \tag{19}$$

Here $p_{n,k}$, $q_{n,k}$, and $s_{n,k}$ are the Fourier coefficients of the following functions:

$$\begin{aligned}
\sqrt{\cosh\xi_0 - \cos\eta}\,(U_k + \epsilon_k V_k) &= \delta_k \sum_{n=0}^{\infty}{}' (p_{n,k}\cos n\eta + \tilde{p}_{n,k}\sin n\eta),\\
-\frac{\epsilon_k V_k}{\sqrt{\cosh\xi_0 - \cos\eta}} &= \epsilon_k \sum_{n=0}^{\infty}{}' (q_{n,k}\cos n\eta + \tilde{q}_{n,k}\sin n\eta),\\
\frac{W_k\sinh\xi_0 - (U_k + \epsilon_k V_k)\sin\eta}{\sqrt{\cosh\xi_0 - \cos\eta}} &= \delta_k \sum_{n=0}^{\infty}{}' (s_{n,k}\sin n\eta - \tilde{s}_{n,k}\cos n\eta),
\end{aligned} \tag{20}$$

and $\kappa_{n,k}$, $\gamma_{n,k}$, and $\lambda_{n,k}$ have the form

$$\kappa_{n,k} = \frac{\delta_k}{\sinh\xi_0\, L^k_{n-\frac{1}{2}(\cosh\xi_0)}}, \quad \gamma_{n,k} = \frac{\varepsilon_k}{L^{k-1}_{n-\frac{1}{2}(\cosh\xi_0)}}, \quad \lambda_{n,k} = \frac{1}{L^{k+1}_{n-\frac{1}{2}(\cosh\xi_0)}}. \tag{21}$$

Coefficients δ_k and ε_k have been introduced to the formulas (20)–(21) to make them accomodate, along with the general asymmetric instances $k \geq 1$, the axisymmetric case $k = 0$:

$$\delta_0 = 1/2, \quad \varepsilon_0 = 0, \quad \delta_k = \varepsilon_k = 1, \quad k \geq 1.$$

Finally, by plugging expressions (19)–(21) into the derived Equations (18), for each harmonic $k \geq 0$, we obtain an infinite system of algebraic equations with respect to variables $x_{n,k}$:

$$a_{n,k}\, x_{n+1,k} - b_{n,k}\, x_{n,k} + c_{n,k}\, x_{n-1,k} = d_{n,k}, \quad n \geq 0, \quad c_{0,k} = 0. \tag{22}$$

The coefficients of the first equation ($n = 0$) in (22) equal to

$$\begin{aligned}
a_{0,k} &= -(2k+1)\kappa_{1,k}\cosh\xi_0 + (1 + 2/\gamma)\kappa_{0,k} - (k+1/2)(k+3/2)\lambda_{1,k} - \gamma_{1,k},\\
b_{0,k} &= (1 + 2/\gamma)\kappa_{0,k}\cosh\xi_0 - (2k+1)\kappa_{1,k} + (k+1/2)^2\lambda_{0,k} - \gamma_{0,k},\\
d_{0,k} &= \gamma_{1,k}\, q_{1,k} - \gamma_{0,k}\, q_{0,k} - (1/2 + 1/\gamma)\kappa_{0,k}\, p_{0,k} - (k+1/2)\kappa_{1,k}\, p_{1,k} - (k+1/2)\kappa_{1,k}\, s_{1,k},
\end{aligned} \tag{23}$$

while for $n \geq 1$ they are

[4] The corresponding formulas for the tilded constants are identical to the above ones, except for the case $n = 0$.

$$
\begin{aligned}
a_{n,k} &= -2(n+k+1/2)\kappa_{n+1,k}\cosh\xi_0 + (2n+1+2/\gamma)\kappa_{n,k} \\
&\quad -(n+k+1/2)(n+k+3/2)\lambda_{n+1,k} - \gamma_{n+1,k}, \\
b_{n,k} &= 2(1+2/\gamma)\kappa_{n,k}\cosh\xi_0 - 2(n+k+1/2)\kappa_{n+1,k} \\
&\quad +2(n-k-1/2)\kappa_{n-1,k} - 2(n+k+1/2)(n-k-1/2)\lambda_{n,k} - 2\gamma_{n,k}, \\
c_{n,k} &= 2(n-k-1/2)\kappa_{n-1,k}\cosh\xi_0 + (-2n+1+2/\gamma)\kappa_{n,k} \\
&\quad -(n-k-1/2)(n-k-3/2)\lambda_{n-1,k} - \gamma_{n-1,k}, \\
d_{n,k} &= \gamma_{n+1,k}\, q_{n+1,k} - 2\gamma_{n,k}\, q_{n,k} + \gamma_{n-1,k}\, q_{n-1,k} \\
&\quad -(n+k+1/2)\kappa_{n+1,k}\, p_{n+1,k} - (1+2/\gamma)\kappa_{n,k}\, p_{n,k} \\
&\quad +(n-k-1/2)\kappa_{n-1,k}\, p_{n-1,k} - (n+k+1/2)\kappa_{n+1,k}\, s_{n+1,k} \\
&\quad +2n\kappa_{n,k}\, s_{n,k} - (n-k-1/2)\kappa_{n-1,k}\, s_{n-1,k}.
\end{aligned} \tag{24}
$$

The infinite system for the variables $\tilde{x}_{n,k}$ associated with tilded constants in (15) has the form

$$
a_{n,k}\,\tilde{x}_{n+1,k} - b_{n,k}\,\tilde{x}_{n,k} + c_{n,k}\,\tilde{x}_{n-1,k} = \tilde{d}_{n,k}, \quad n \ge 1, \quad \tilde{x}_{0,k} = 0, \tag{25}
$$

where $\tilde{d}_{n,k}$, $n \ge 1$ have the form similar to $d_{n,k}$ in (24):

$$
\begin{aligned}
\tilde{d}_{n,k} &= \gamma_{n+1,k}\,\tilde{q}_{n+1,k} - 2\gamma_{n,k}\,\tilde{q}_{n,k} + \gamma_{n-1,k}\,\tilde{q}_{n-1,k} - (n+k+1/2)\kappa_{n+1,k}\,\tilde{p}_{n+1,k} \\
&\quad -(1+2/\gamma)\kappa_{n,k}\,\tilde{p}_{n,k} + (n-k-1/2)\kappa_{n-1,k}\,\tilde{p}_{n-1,k} \\
&\quad -(n+k+1/2)\kappa_{n+1,k}\,\tilde{s}_{n+1,k} + 2n\kappa_{n,k}\,\tilde{s}_{n,k} - (n-k-1/2)\kappa_{n-1,k}\,\tilde{s}_{n-1,k},
\end{aligned}
$$

provided that $\tilde{q}_{0,k} = \tilde{p}_{0,k} = 0$.

Reduction of the displacement boundary-value problem for a torus to the three-diagonal infinite systems (22), (25) represents an exact solution of the boundary-value problem (1)–(2) in terms of the solvability of the basic potential-theory problems in toroidal regions. Indeed, it can be demonstrated that the Neumann problem for the equation $\Delta_k(\cdot) = 0$ in a toroidal region reduces to equations of type (22), implying that harmonics in a toroidal-region function cannot be determined simpler than by a set of tridiagonal infinite systems, unless its value is specified at the whole boundary.[5]

Given that the displacement boundary-value problem of elasticity cannot be reduced to pure Dirichlet problems for harmonic functions, the obtained tridiagonal infinite systems of algebraic equations (22), (25) qualify as an exact solution of the boundary-value problem (1)–(2) for a torus.

We strengthen this result by presenting analytical techniques for solving the tridiagonal systems of type (22), thereby finalizing the construction of an exact analytical solution of the displacement boundary-value problem of elasticity for a torus.

[5] The Dirichlet problem for a torus has an exact solution, *i.e.*, the coefficients of the harmonic function are determined directly as the Fourier transforms of its boundary value. The Neumann problem for a torus features a similar solution in the axisymmetric case *only*, due to the fact that for $k = 0$ the corresponding tridiagonal system can be split into two bidiagonal ones.

4. Solution of infinite algebraic systems of diagonal form

An analytical technique for solving systems of type (22) with three variables in a row was suggested by Kutsenko and Ulitko [12]. According to [12], the original tridiagonal infinite system of algebraic equations

$$a_n\, x_{n+1} - b_n\, x_n + c_n\, x_{n-1} = d_n, \quad c_0 = 0, \quad n \ge 0, \tag{26}$$

can be reduced to successively solving two bidiagonal systems

$$\frac{\bar a_n}{1-\beta_n}\, x_{n+1} - x_n = Y_n, \quad n \ge 0; \quad Y_n - \frac{\bar c_n}{1-\beta_n}\, Y_{n-1} = \frac{\bar d_n}{1-\beta_n}, \quad n \ge 1, \tag{27}$$

where $\bar a_n = a_n/b_n$, $\bar c_n = c_n/b_n$, $\bar d_n = d_n/b_n$, and β_n is a finite continuous fraction:

$$\beta_n = \frac{\bar c_n \bar a_{n-1}}{1-\beta_{n-1}}, \quad \beta_0 = 0, \quad \text{or} \quad \beta_n = \cfrac{\bar c_n \bar a_{n-1}}{1 - \cfrac{\bar c_{n-1}\bar a_{n-2}}{1 - \cdots \, \ddots \, 1 - \cfrac{\bar c_1 \bar a_0}{1-0}}}.$$

The above bidiagonal systems have analytical solutions

$$x_n = -Y_n - \sum_{k=n+1}^{\infty} Y_k \prod_{s=n}^{k-1} \frac{\bar a_s}{1-\beta_s}, \quad n \ge 0,$$
$$Y_n = \frac{\bar d_n}{1-\beta_n} + \sum_{k=0}^{n-1} \frac{\bar d_k}{1-\beta_k} \prod_{s=k+1}^{n} \frac{\bar c_s}{1-\beta_s}, \quad n \ge 0.$$

We present a different approach that can be applied to systems with a greater number of variables in a row. To illustrate the idea on the tridiagonal system (26) let us try a solution of (22) in the form

$$x_n = \alpha_n \sum_{j=n}^{\infty} y_j, \quad n \ge 0, \tag{28}$$

where y_j are the new unknowns, and the coefficients α_n are undetermined so far. Substituting (28) in (26) we have

$$(a_n\alpha_{n+1} - b_n\alpha_n + c_n\alpha_{n-1}) \sum_{j=n+1}^{\infty} y_j - (b_n\alpha_n - c_n\alpha_{n-1}) y_n + c_n\alpha_{n-1} y_{n-1} = d_n, \quad n \ge 0.$$

If one requires the coefficients α_n^k to satisfy the conditions

$$a_n\alpha_{n+1} - b_n\alpha_n + c_n\alpha_{n-1} = 0, \quad n \ge 0, \quad c_0 = 0, \tag{29}$$

then y_n will be determined from the bidiagonal system

$$a_n\alpha_{n+1} y_n - c_n\alpha_{n-1} y_{n-1} = -d_n, \quad n \ge 0, \quad c_0 = 0, \tag{30}$$

whose exact solution is

$$y_n = -\frac{1}{a_n \alpha_{n+1} \alpha_n} \left\{ \alpha_n\, d_n + \sum_{j=0}^{n-1} \alpha_j\, d_j \prod_{i=j}^{n-1} \frac{c_{i+1}}{a_i} \right\}, \quad n \geq 0. \tag{31}$$

To obtain a solution of the homogeneous equations (29) that involves only one arbitrary constant, we rewrite (29) in the form

$$a_0 \alpha_1 = b_0 \alpha_0^*, \quad a_1 \alpha_2 - b_1 \alpha_1 = -c_1 \alpha_0^*, \quad a_n \alpha_{n+1} - b_n \alpha_n + c_n \alpha_{n-1} = 0, \quad n \geq 2, \tag{32}$$

where $\alpha_0^* =$ const $\neq 0$. Then, solution of (29) can be obtained by recursion:

$$\alpha_{-1} = 0, \quad \alpha_0 = \alpha_0^*, \quad \alpha_n = \frac{1}{a_{n-1}} (b_{n-1} \alpha_{n-1} - c_{n-1} \alpha_{n-2}), \quad n \geq 1. \tag{33}$$

Note that constant α_0^* may be assigned an arbitrary non-zero value, since it cancels out after substituting (33) in the expressions (31).

The formal proof of convergence for the presented solutions requires the use of theories of continued fractions and sequence transformations, and is beyond the scope of this paper. Nevertheless, convergence in formulas (28)–(33) can be verified in every particular case by the analysis of the coefficients of system (26). For example, for the problems involving the torus' exterior, such as the Stokes problem for a torus considered in Section 6, the coefficients of the systems (22), (25) converge exponentially to zero, making their solutions converge as well.

5. On the uniqueness of the solutions for the boundary-value problems of elasticity in toroidal regions

Besides incomplete separation of variables in toroidal coordinates, the displacement boundary-value problem for a torus features one more challenge in constructing the solution.

The problem is that certain classes of vector boundary-value problems may not be uniquely solved in multiply-connected regions. In particular, vector boundary-value problems that involve the Cauchy-Riemann equations require additional analysis of the single-valuedness of their solutions. To such a class of problems belong the boundary-value problems of elasticity, where the elastostatics equation (1) ‘contains’ the so-called *generalized Cauchy-Riemann equations*.

To show that solving the elastostatics equations always involves solving equations of Cauchy-Riemann type, we write (1) as two consecutive systems of the fundamental equations for vector fields

$$\begin{cases} \operatorname{curl} \boldsymbol{\omega} = -\operatorname{grad} \vartheta, \\ \operatorname{div} \boldsymbol{\omega} = 0, \end{cases} \qquad \begin{cases} \operatorname{curl} \mathbf{u} = \boldsymbol{\omega}, \\ \operatorname{div} \mathbf{u} = -\dfrac{1-2\nu}{2(1-\nu)} \vartheta. \end{cases} \tag{34}$$

Each pair of equations in (34) restores a vector field by its vorticity and divergence. Along with the Lamé equation, systems (34) represent an invariant form of the equations for equilibrium of an elastic medium, but in contrast to (1), they emphasize the structure of the displacement and vorticity fields in an elastic solid.

By representing, similarly to (7), the vorticity vector $\boldsymbol{\omega}$ by its Fourier series with components ω_{rk}, $\omega_{\varphi k}$, and ω_{zk}, from the first system of (34), one can derive two sets of generalized Cauchy-Riemann equations

$$\left(\frac{\partial}{\partial r} \mp \frac{k \mp 1}{r}\right) \Omega_k^\pm = -\frac{\partial \Theta_k^\pm}{\partial z}, \quad \frac{\partial \Omega_k^\pm}{\partial z} = \left(\frac{\partial}{\partial r} \pm \frac{k}{r}\right) \Theta_k^\pm, \tag{35}$$

where the functions $\Omega_k^\pm$ and $\Theta_k^\pm$ contain the Fourier components of vorticity $\boldsymbol{\omega}$ and 'dilatation' ϑ:

$$\Omega_k^\pm = \omega_{rk} \pm \omega_{\varphi k}, \quad \Theta_k^\pm = \omega_{zk} \pm \vartheta_k.$$

Note that at infinity ($r, z \to \infty$) both '+' and '−' systems (35) become identical to the classical Cauchy-Riemann equations governing the real and imaginary parts of an analytic function of a complex variable [1]. This observation makes clear the analogy between (35) and the regular Cauchy-Riemann equations, and supports the conjecture that solutions of both types of systems should reveal similar properties. Namely, in the same way that analytic functions must satisfy some predetermined constraints to be single-valued in a multiple-connected region, the functions $\Omega_k^\pm$ and $\Theta_k^\pm$ have to satisfy certain conditions to be unique solutions to the Equations (35).[6]

To show this, first observe that the functions $\Omega_k^\pm$, $\Theta_k^\pm$ in (35) must satisfy the equations

$$\Delta_{k\mp 1}\Omega_k^\pm = 0, \quad \Delta_k \Theta_k^\pm = 0,$$

and therefore can be presented in toroidal coordinates as

$$\begin{aligned}\Omega_k^\pm &= \sqrt{\cosh\xi - \cos\eta}\, {\sum_{n=0}^{\infty}}' (X_{n,k}^\pm \sin n\eta - \tilde{X}_{n,k}^\pm \cos n\eta) L_{n-\frac{1}{2}}^{k\mp 1}(\cosh\xi),\\ \Theta_k^\pm &= \sqrt{\cosh\xi - \cos\eta}\, {\sum_{n=0}^{\infty}}' (Y_{n,k}^\pm \cos n\eta + \tilde{Y}_{n,k}^\pm \sin n\eta) L_{n-\frac{1}{2}}^{k}(\cosh\xi).\end{aligned} \tag{36}$$

Then, to ensure the single-valuedness of the functions (35) one must require their coefficients to satisfy the following conditions:

$$ {\sum_{n=0}^{\infty}}' Y_{n,k}^+ = 0, \quad {\sum_{n=0}^{\infty}}' Y_{n,k}^- \prod_{j=1}^{k}\Big[n^2 - (j-1/2)^2\Big] = 0, \tag{37}$$

(as before, we make our arguments for the untilded coefficients). *There are no constraints for the series* $X_{n,k}^\pm$. The physical reasoning behind relations (37) can be easily demonstrated for the torus exterior. Let the functions $\Omega_k^\pm$ be expressed in terms of $\Theta_k^\pm$ from Equations (35) in the following way:

$$r^{\mp k+1}\Omega_k^\pm = \int_L r^{\mp k+1}\left\{-\frac{\partial \Theta_k^\pm}{\partial z}\mathrm{d}r + \left(\frac{\partial \Theta_k^\pm}{\partial r} \pm \frac{k}{r}\Theta_k^\pm\right)\mathrm{d}z\right\}, \tag{38}$$

[6]The single-valuedness of the solutions of equations of type (35) is closely connected to the so-called *generalized Dirichlet problem* for analytic functions in the complex plane [13, pp. 367–390], [14, pp. 163–186]. This problem consists in finding a harmonic function by its boundary value, subject to the condition that its conjugate by the Cauchy-Riemann conditions counterpart must be single-valued in the same region. In [13, pp. 367–390], [14, pp. 163–186] it was demonstrated that the solution of the generalized Dirichlet problem for a planar single-connected region exists and is unique. However, for the existence of the solution in multiply-connected planar regions some predetermined constraints must be imposed on the boundary value of the sought harmonic function. Evidently, this result should also hold for the generalized Cauchy-Riemann equations (35).

where L is an arbitrary curve in the cross-sectional half-plane $\{r \geq 0 \mid \varphi = \text{ const}\}$. If one wants the functions $\Omega_k^\pm$ determined by (38) to be single-valued in a double-connected region, then the integrals (38) must vanish on an arbitrary closed contour L lying in the region. Taking L as a circle $\xi = \xi_1$ enclosing the torus and doing the integration, we obtain the relations (37).

Another way to derive the constraints (37) is to use the recursive relations between the Legendre functions of adjacent indices [11, pp. 160–162]. Since these relations are identical for both functions $P^k_{n-\frac{1}{2}}(\cosh\xi)$ and $Q^k_{n-\frac{1}{2}}(\cosh\xi)$, equalities (37) must hold in the interior, as well as in the exterior of a torus.

Similarly, it can be shown that *the coefficients* $Y^\pm_{n,k}$ *are always uniquely determined by known the constants* $X^\pm_{n,k}$. This 'asymmetry' in the solvability of the generalized Cauchy-Riemann equations in toroidal domains is caused by the presence of the Riemannian radical in the expressions (36) for the functions $\Omega_k^\pm$ and $\Theta_k^\pm$.

Now we show that the introduced form (5) of the general solution of the elastostatics equations ensures the uniqueness of the solution of the boundary-value problem (1)–(2) in toroidal regions. Indeed, in terms of functions (15) conditions (37) become

$$\begin{aligned}(k\gamma+1)\sum_{n=0}^{\infty}{}' A_{n,k} &= (2k^2+k)\sum_{n=0}^{\infty}{}' B_{n,k},\ (2-2k\gamma)\sum_{n=0}^{\infty}{}' A_{n,k}\prod_{j=1}^{k}\Big[n^2-(j-1/2)^2\Big] \\ &= (4k^2-2k)\sum_{n=0}^{\infty}{}' D_{n,k}\prod_{j=1}^{k-1}\Big[n^2-(j-1/2)^2\Big].\end{aligned} \tag{39}$$

Then, on summing Equations (18) multiplied by the factors 1 and $\prod_{j=1}^{k}\Big[n^2-(j-1/2)^2\Big]$, respectively, we find that conditions (39) turn into identities, if constants $A_{n,k}$, $B_{n,k}$, $C_{n,k}$, and $D_{n,k}$ satisfy Equations (18).

Let us emphasize again that the identical fulfillment of (37), and consequently, the uniqueness of the obtained solution of the displacement boundary-value problem for a torus are ensured by the specific form of the general solution (5)–(6). Other forms of general solution may require additional argumentation for satisfying the conditions of type (37). An example of a general solution that yields a non-unique solution of a vector boundary-value problem in a toroidal region is presented in the next section within the scope of the Stokes problem for a torus.

6. Example: The Stokes problem for a torus

The comprehensiveness of the developed solution for the displacement boundary-value problem for a torus makes the author reluctant to illustrate the approach by putting some numbers into the general formulas. Ultimately, such an example will be no more than an exercise in developing the Fourier expansions (2), (20) for the boundary conditions, with subsequently plugging the calculated Fourier coefficients into the right-hand side of Equations (22). Instead, we invite the reader to revisit a classical problem of axisymmetric motion of a rigid torus in an unbounded Stokes flow. We will show how the elastic solutions may be applied to the problems of viscous incompressible flows, and discuss two different approaches to solving the Stokes problem for a torus. One of them uses the presented elasticity solution, while the other will illuminate the ideas of Section 5. We will also demonstrate the relationship between these two approaches, and how one of them can be used to derive an elegant solution for the other.

The most significant contribution to the Stokes problem for a torus can be found, in our opinion, in the papers by Pell and Payne [7] and Wakiya [8] that feature the two mentioned approaches to the construction of the solution. However, it was the unanswered questions left in both these papers that, in fact, stimulated our initial interest in the presented work. Here we present a *complete* solution to the Stokes problem for a torus, and demonstrate how the quite different approaches of [7] and [8] relate to each other.

6.1. Motion of a torus in a viscous Stokes flow

The term 'Stokes problem' with respect to a particular body usually addresses the problem of a slow, steady motion of that body in a viscous incompressible fluid, or, equivalently, a problem of steady viscous flow about the body [6]. The name attributes to G. G. Stokes, who first formulated and solved the problem of the motion of a rigid sphere in a viscous fluid [15].

We will employ the general elasticity solution developed above to solve the Stokes problem for a torus. Given that the Stokes problem is essentially a hydromechanics problem, this kind of approach can be justified by means of the *analogy* between the boundary-value problems of elastostatics and those of steady Stokes flows of viscous fluid that we present below.

The Stokes equations

$$\operatorname{curl}\operatorname{curl}\mathbf{u} = -\frac{1}{\mu}\operatorname{grad} p, \quad \operatorname{div}\mathbf{u} = 0, \tag{40}$$

describe *slow* flows of viscous incompressible fluids [6, pp. 58–62]. Here $\mathbf{u}$ is the velocity vector, p is the hydrostatic pressure, and μ is the dynamic viscosity coefficient. Equations (40) represent a linearized form of the general Navier-Stokes equations in the approximation of low Reynolds number [6, pp. 23–29, 40–42]. The second equation of (40) is the *continuity* equation that prescribes the fluid incompressibility. Writing the Stokes equations in a form similar to (34):

$$\begin{cases} \operatorname{curl}\boldsymbol{\omega} = -\frac{1}{\mu}\operatorname{grad} p, \\ \operatorname{div}\boldsymbol{\omega} = 0, \end{cases} \qquad \begin{cases} \operatorname{curl}\mathbf{u} = \boldsymbol{\omega}, \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{41}$$

and setting in (34) $\nu = 1/2$ (*i.e.*, medium is incompressible) and $\vartheta = p/\mu$, we observe that both the Lamé equation (1) and the Stokes equations (40) reduce to the same system of equations for the vector fields in (41). Hence, equilibrium of an elastic incompressible solid and steady flow of a Stokes fluid are governed by the same equations. In the framework of the Stokes problem $\mathbf{u}$ and μ represent the velocity vector and dynamic viscosity coefficient, whereas in elasticity they stand for the elastic displacement and shear modulus respectively.[7]

So, consider the problem of determining the velocity and pressure fields in a viscous incompressible flow due to a steady translation of a rigid toroidal body along its symmetry axis, which is assumed to coincide with the axis Oz of the cylindrical coordinates. Besides being a solution of the Stokes equations (40), $\mathbf{u}$ and p must satisfy the boundary conditions that in the cylindrical coordinates have the form [6, p. 29]

$$u_r|_S = 0, \quad u_z|_S = V; \qquad p|_\infty = p_\infty, \tag{42, 43}$$

[7]The considered analogy between elasticity and Stokes flow problems does not span the dynamic problems. Indeed, note that the acceleration term enters the Lamé equation as $\partial^2\mathbf{u}/\partial t^2$, whereas in the dynamic Stokes equations it reads as $\partial\mathbf{u}/\partial t$.

where V is the translation velocity, an S is the torus's surface. Equalities (42) reflect the general property of viscous flows, *i.e.*, adherence of the fluid particles to the body's surface. And without loss of generality, the constant hydrostatic pressure at infinity p_∞ is assumed to be zero.

Below we demonstrate how the general theory considered in the previous sections can be used to construct a complete and comprehensive solution for the Stokes problem (40), (42) for a torus. In view of the presented analogy between problems of elastostatics and Stokes flows, one can obtain the solution for the Stokes problem for a torus by adapting the developed general solution of the displacement boundary-value problem of elasticity for toroidal bodies. However, we will defer the application of this approach for a few pages, and first consider the traditional method for solving the Stokes-flow problems, the so-called *stream function* approach. After that we will show how these two approaches relate to each other, and how to use one of them to construct the solution for the other.

6.2. Solution of the Stokes problem for a torus using stream function

The traditional approach [6, pp. 96–100] to solving the axisymmetric Stokes equations is to represent the velocity vector **u** as[8]

$$u_r = -\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial}{\partial r}(r\Psi)\,. \tag{44}$$

This representation satisfies identically the second equation in (40); the first equation yields the condition to be satisfied by function Ψ:

$$\Delta_1 \Delta_1 \Psi = 0. \tag{45}$$

In terms of function Ψ the boundary conditions (42) become

$$\Psi|_S = \frac{V}{2}\left(r + \frac{\lambda}{r}\right)\bigg|_S, \quad \frac{\partial \Psi}{\partial n}\bigg|_S = \frac{V}{2}\frac{\partial}{\partial n}\left(r + \frac{\lambda}{r}\right)\bigg|_S, \tag{46}$$

where n is the normal to the surface S of the torus. The unknown constant λ attributes to the multiple-connectedness of torus surface, and has to be determined from the solution of the problem. For single-connected bodies, λ is always zero. It turns out that to determine the value of λ, one has to consider the solvability problem for the generalized Cauchy-Riemann equations. Failure to determine the correct value of λ leads to physically inadequate results, as in [16].

The explicit form of function Ψ in the toroidal coordinates is required to solve the boundary-value problem (46). First, observe that Ψ, as a general solution of Equation (45), admits the next representation by two harmonic functions:

$$\Psi = \Phi_1 + rP, \qquad \Delta_1 \Phi_1 = 0, \quad \Delta P = 0. \tag{47}$$

Taking into consideration that the problem at hand is an exterior problem, and omitting the coefficients that will cancel out after satisfying the boundary conditions (46), we write the representations for the functions Φ_1 and P in the toroidal coordinates as

[8]Function Ψ defined by (44) differs from the from the classical *stream function* ψ introduced in [6, pp. 96–100]: $\Psi = -\psi/r$. The choice of either function in representation of type (44) is merely a matter of convenience.

$$\begin{aligned} \Phi_1 &= \sqrt{\cosh\xi - \cos\eta}\;{\sum_{n=0}^{\infty}}' \varpi_n \cos n\eta\, P^1_{n-\frac{1}{2}}(\cosh\xi), \\ P &= \frac{1}{c}\sqrt{\cosh\xi - \cos\eta}\;{\sum_{n=0}^{\infty}}' \varrho_n \cos n\eta\, P_{n-\frac{1}{2}}(\cosh\xi). \end{aligned} \tag{48}$$

Combining (48) and (47), we obtain the general form of function Ψ in the toroidal coordinates:

$$\begin{aligned} \Psi = &\frac{1}{2\sqrt{\cosh\xi - \cos\eta}} \Bigg\{ g_0 P^1_{\frac{1}{2}}(\cosh\xi) + h_0 \cosh\xi\; P^1_{-\frac{1}{2}}(\cosh\xi) \\ &+ \sum_{n=1}^{\infty} \Big(g_n P^1_{n-\frac{3}{2}}(\cosh\xi) + h_n P^1_{n+\frac{1}{2}}(\cosh\xi) \Big) \cos n\eta \Bigg\}, \end{aligned} \tag{49}$$

where coefficients g_n and h_n of function Ψ are expressed in terms of the coefficients of Φ_1 and P as

$$\begin{aligned} g_0 &= 2\varrho_0 - \varpi_1, \quad h_0 = \varpi_0 - 2\varrho_0, \\ g_n &= \frac{n+1/2}{n}\,\varpi_n - \varpi_{n-1} - \frac{\varrho_n}{n}, \quad n \geq 1, \\ h_n &= \frac{n-1/2}{n}\,\varpi_n - \varpi_{n+1} + \frac{\varrho_n}{n}, \quad n \geq 1. \end{aligned} \tag{50}$$

Substituting the representation (49) in both equalities of the boundary conditions (46), we obtain a separate system of two linear algebraic equations with respect to g_n and h_n for each $n \geq 0$. The right-hand sides of these equations are linear in λ (recall that the unknown constant λ enters the right-hand sides of the boundary conditions (46) linearly), therefore coefficients g_n and h_n of function Ψ have the form

$$g_n = g_n^{(1)} + \frac{\lambda}{c^2}\, g_n^{(2)}, \quad h_n = h_n^{(1)} + \frac{\lambda}{c^2}\, h_n^{(2)}, \tag{51}$$

where the explicit expressions for $g_n^{(1,2)}$ and $h_n^{(1,2)}$ are furnished in the Appendix.

The value of λ in (51) *cannot be determined from the boundary-value conditions* (46). Instead, the choice of λ is justified by the uniqueness of the solution of the boundary-value problem (42), *i.e.*, the velocity vector $\mathbf{u}$ as well as the hydrostatic pressure p must be single-valued in the flow region.

According to the discussion in the preceding section, the unique solution of elastostatics boundary-value problems in toroidal regions is achieved by imposing conditions (37) on the components of vorticity and dilatation in the medium. In the context of the Stokes problem for a torus, these conditions acquire a transparent physical meaning. Indeed, since the velocity components u_r and u_z are uniquely determined by the function Ψ (44), then for any value of λ in (51) they are single-valued functions of the spatial coordinates. However, the hydrostatic pressure p cannot be directly determined from the solution of the boundary-value problem (46) for function Ψ. To find $p = \vartheta\mu$, one has to solve the generalized Cauchy-Riemann equations (35), whose unique solution is guaranteed by the conditions of type (37).

Given the axial symmetry of the considered problem, Equations (35) reduce to a single system governing vorticity ω and pressure $\vartheta = p/\mu$ in the fluid:

$$\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\omega=-\frac{\partial\vartheta}{\partial z},\quad \frac{\partial\omega}{\partial z}=\frac{\partial\vartheta}{\partial r}. \tag{52}$$

Simplifying the notations (36), we present $\omega=\Omega_0^{\pm}$ and $p/\mu=\vartheta=\Theta_0^{\pm}$ as

$$\begin{aligned}\omega &= -\frac{1}{c^2}\sqrt{\cosh\xi-\cos\eta}\,{\sum_{n=0}^{\infty}}'\omega_n\cos n\eta\, P^1_{n-\frac{1}{2}}(\cosh\xi),\\ p &= \frac{\mu}{c^2}\sqrt{\cosh\xi-\cos\eta}\sum_{n=1}^{\infty}p_n\sin n\eta\, P_{n-\frac{1}{2}}(\cosh\xi).\end{aligned} \tag{53}$$

With regard to functions (53), the uniqueness condition of type (37) reads:

$$\sum_{n=0,1}^{\infty}\omega_n=0. \tag{54}$$

Then, from (44) it follows that $\omega=\dfrac{\partial u_r}{\partial z}-\dfrac{\partial u_z}{\partial r}=-\Delta_1\Psi$, and after somewhat tedious but straightforward transformations we find

$${\sum_{n=0}^{\infty}}'\omega_n=-\frac{3}{2}\sum_{n=0}^{\infty}(g_n+h_n). \tag{55}$$

Finally, relationships (54) and (55) yield an expression for λ:

$$\frac{\lambda}{c^2}=-\sum_{n=0}^{\infty}\left(g_n^{(1)}+h_n^{(1)}\right)\Bigg/\sum_{n=0}^{\infty}\left(g_n^{(2)}+h_n^{(2)}\right). \tag{56}$$

The Fourier coefficients p_n of the hydrostatic pressure p can be restored by the known vorticity coefficients ω_n using the equations (52)[9]:

$$p_n=-(n-1/2)\omega_n+\sum_{j=n+1}^{\infty}\omega_j=-(n-1/2)\omega_n-{\sum_{j=0}^{n}}'\omega_j.$$

An expression of type (56) was also derived in the paper by Pell and Payne [7], but their argumentation was rather vague. Section 5 of the present paper justifies the condition (54) and the subsequent expression (56) from the general viewpoint of solvability of the equations for vector fields and the generalized Cauchy-Riemann equations in the double-connected toroidal regions.

6.3. APPLICATION OF THE GENERAL ELASTICITY SOLUTION TO THE STOKES PROBLEM

An alternative way to solve the considered Stokes problem for a torus is to employ the general solution (10) for the axisymmetric case ($k=0$):

$$u_r=\frac{1}{2}u_0,\quad u_z=\frac{1}{2}w_0, \tag{57a}$$

[9] Equations for coefficients ω_n and p_n are derived in a way similar to the derivation of Equations (18). Each of the Equations (52) yields an infinite system of linear algebraic equations that is bidiagonal either in terms of ω_n or p_n, and therefore can be solved explicitly.

where in expressions (10) for u_0 and w_0 we set $\nu = 1/2$ and $\vartheta = \vartheta_0/2 = p/\mu$:

$$u_0 = \phi_1 + \frac{r}{\mu} p, \quad w_0 = \chi_0 + \frac{z}{\mu} p. \tag{57b}$$

The functions p, ϕ_1, and χ_0 are not independent, and have to satisfy the differential constraint (11) with $k = 0$:

$$\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\frac{p}{\mu} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\phi_1 + \frac{\partial \chi_0}{\partial z} = 0, \tag{58}$$

which is a rephrase of the continuity equation (40)[10]. The boundary conditions (42) in terms of the functions (57) take the form

$$\left.\frac{p}{\mu}\right|_{\xi=\xi_0} = -\left.\frac{1}{r}\phi_1\right|_{\xi=\xi_0}, \quad \chi_0|_{\xi=\xi_0} = \left.\left(\frac{z}{r}\phi_1 + 2V\right)\right|_{\xi=\xi_0}, \tag{59}$$

where $\xi = \xi_0$ is the surface of the torus. Representing the functions (57) in accordance to (15),

$$\begin{aligned} \frac{2p}{\mu} &= \frac{1}{c}\sqrt{\cosh\xi - \cos\eta}\sum_{n=1}^{\infty} \tilde{A}_{n,0}\ \sin n\eta\ P_{n-\frac{1}{2}}(\cosh\xi), \\ \phi_1 &= \sqrt{\cosh\xi - \cos\eta}\sum_{n=1}^{\infty} \tilde{B}_{n,0}\ \sin n\eta\ P^1_{n-\frac{1}{2}}(\cosh\xi), \\ \chi_0 &= -\sqrt{\cosh\xi - \cos\eta}\sum_{n=0}^{\infty}{}' \tilde{C}_{n,0}\ \cos n\eta\ P_{n-\frac{1}{2}}(\cosh\xi), \end{aligned} \tag{60}$$

and repeating steps (18)–(20), we come to a tridiagonal system with respect to variables $\tilde{x}_{n,0}$[11]:

$$a_{n,0}\,\tilde{x}_{n+1,0} - b_{n,0}\,\tilde{x}_{n,0} + c_{n,0}\,\tilde{x}_{n-1,0} = \tilde{d}_{n,0}, \quad n \geq 1, \quad \tilde{x}_{0,0} = 0, \tag{61}$$

where in the expression (24) for coefficients $a_{n,0}$, $b_{n,0}$, and $c_{n,0}$ it must be set $\gamma = 1/2$, and the right-hand side $\tilde{d}_{n,0}$ contains only terms with $\tilde{s}_{n,0}$:

$$\tilde{s}_{n,0} = -\frac{8\sqrt{2}}{\pi} V\ \sinh\xi_0\ Q_{n-\frac{1}{2}}(\cosh\xi_0), \quad n \geq 1. \tag{62}$$

The exact solution of (61)–(62) can be obtained as shown in (28)–(33).

Note that by plugging in (57)–(62) a value of the coefficient γ other than $1/2$, we obtain the solution of the elasticity problem where a rigid toroidal inclusion in an elastic space is subject to a vertical shift of magnitude V.

6.4. Analysis of the flow around a torus and numerical results

In this subsection we present numerical results for the Stokes problem for a torus, using both solutions presented in the Subsections 6.2 and 6.3. But first we have to discuss the numerical procedures used for computation of the Legendre functions of the first kind $P^k_{n-\frac{1}{2}}(\cosh\xi)$ and expansions of type (15).

---

[10] Note that the first equation of (40) is satisfied identically by the representation (57).

[11] Formulas (18) to (24) were developed for the ‘untilded’ coefficients in (15). Analogous expressions for the ‘tilded’ ones have almost the same form, differing in representations for $n = 0$.

According to the asymptotic behavior of the Legendre functions [11, pp. 163–164], functions $P^k_{n-\frac{1}{2}}(\cosh\xi)$ grow with n and ξ as $e^{n\xi}$. Therefore, the use of the integral representation (16a) for calculating of the Legendre functions may lead to significant inaccuracies, even at small n. An acceptable degree of accuracy in computing the Legendre functions $P^{(0,1)}_{n-\frac{1}{2}}(\cosh\xi)$, which are used in the solution of the Stokes problem for a torus, is achieved by implementing the following recursive scheme:

$$
\begin{aligned}
P_{n+\frac{1}{2}}(\cosh\xi) &= \cosh\xi\, P_{n-\frac{1}{2}}(\cosh\xi) + \frac{\sinh\xi}{n+\frac{1}{2}}\, P^1_{n-\frac{1}{2}}(\cosh\xi), \quad n \geq 0,\\
P^1_{n+\frac{1}{2}}(\cosh\xi) &= \cosh\xi\, P^1_{n-\frac{1}{2}}(\cosh\xi) + (n+\frac{1}{2})\sinh\xi\, P_{n-\frac{1}{2}}(\cosh\xi), \ n \geq 0,
\end{aligned}
$$

where the functions $P^{(0,1)}_{-\frac{1}{2}}(\cosh\xi)$ are computed using the corresponding integral representations. Note that due to the exponential growth in n of the functions $P^k_{n-\frac{1}{2}}(\cosh\xi)$, the coefficients in the expansions of type (48), (49), (60) must converge at least exponentially to ensure the convergence of these series, and, as a result, such expansions also exhibit exponential convergence. Therefore, it is usually sufficient to retain about 20–25 terms in a series of type (48), (49), (60) to achieve the degree of accuracy within $10^{-6} \div 10^{-8}$. To compute the solution of the tridiagonal system (61), using the technique (28)–(33), we truncated the infinite series (31) at $n = 50$.

Let us now proceed with the analysis of the viscous flow due to the translational motion of a rigid torus in a fluid.

First, we compute the drag force exerted on the surface of the torus, which is the major integral characteristic of the flow about a body. The drag force is obtained by integrating the surface tractions over the torus's surface. Due to the axial symmetry of the problem, the only non-zero component of the drag force is F_z:

$$
\begin{aligned}
F_z &= -2\sqrt{2}\pi\mu\left\{-h_0 + \sum_{n=1}^{\infty}\left[(n+\tfrac{3}{2})(n+\tfrac{1}{2})g_n + (n-\tfrac{1}{2})(n-\tfrac{3}{2})h_n\right]\right\}\\
&= -2\sqrt{2}\pi\mu c \sum_{n=0}^{\infty}{}' (n\tilde{A}_n - 2\tilde{C}_n).
\end{aligned} \tag{63}
$$

The graph of the drag force F_z, normalized by the drag force of a sphere with radius a equal to the radius of the outermost rim of the torus, is presented in Figure 2. With ξ_0 approaching zero, the normalized drag force (63) reaches the limiting value of 0·935, which is the normalized drag force for the so-called 'closed torus', *i.e.* a torus without opening [17]. When $\xi_0 \to \infty$, and torus degenerating into a thin circular thread, the asymptotic value of the normalized drag force equals to

$$
\frac{F_z}{6\pi\mu Va} \sim \frac{4\pi}{3\xi_0}, \quad \xi_0 \to \infty,
$$

which is the normalized drag force for a thin annulus.

The flux through the torus opening is characterized by the value of constant the λ (56):

$$
Q = -2\pi\int_0^b (u_z - V)\, r\, \mathrm{d}r = -\pi V\lambda \tag{64}
$$

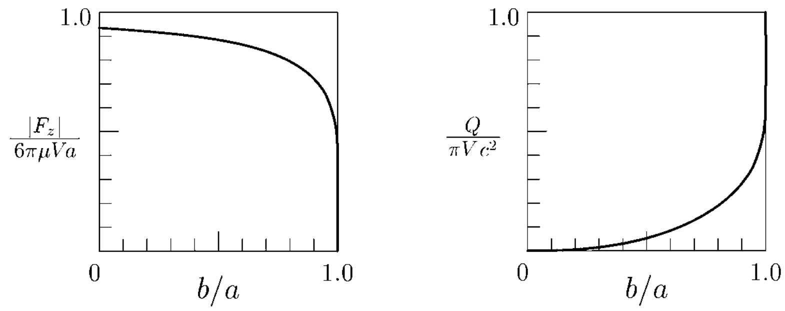

Figure 2. Normalized drag force exerted on the torus and the flux through the torus opening.

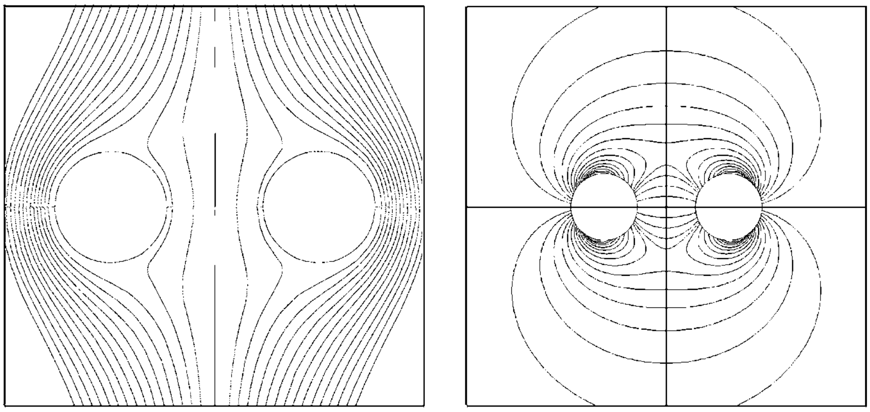

Figure 3. Streamlines and isobars of a viscous flow in the vicinity of a torus ($b/a = 0{\cdot}3$).

The graph of the flux Q is shown in Figure 2b. When ξ_0 approaches infinity, Q becomes equal to the flux of an unperturbed flow through a circle of radius a. For a closed torus ($\xi_0 \sim 0$), the flux Q is obviously zero.

Figures 3a and 3b display the patterns of streamlines and isobars of a viscous flow in the vicinity of a torus with radii ratio $b/a = 0{\cdot}3$. The streamlines are determined as the solutions to the equation

$$\psi = -r\Psi = \text{ const},$$

where ψ is the Stokes stream function [6, pp. 94–100]. However, from a numerical point of view it is better to construct the streamlines as the solutions to the equation

$$\frac{\mathrm{d}r}{u_r} = \frac{\mathrm{d}z}{u_z - V_0}$$

In the same way, the lines of constant pressure $p =$ const can be obtained from the equation

$$\frac{\mathrm{d}r}{\dfrac{\partial\omega}{\partial r} + \dfrac{\omega}{r}} = \frac{\mathrm{d}z}{\dfrac{\partial\omega}{\partial z}}$$

which is derived from the generalized Cauchy-Riemann equations (52).

As the final step in solving the Stokes problem for a torus, we demonstrate that both general solutions (44) and (57) of the Stokes equations, though being derived in different ways, are closely related. Moreover, this relationship can be used to obtain an explicit solution of the boundary-value problem (57), (59) without solving the tridiagonal system (61).

6.5. Relationship between solutions (44) and (57)

It is possible to construct an exact explicit solution of system (61) without resorting to the techniques presented in Section 4. Instead, one may employ the obtained solution of the boundary-value problem (46) for function Ψ to solve Equations (61).

First, observe that Φ_1 in (47) and ϕ_1 in (57), as the solutions of equation $\Delta_1(\cdot) = 0$, can be represented by derivatives of some harmonic functions Φ and ϕ:

$$\Phi_1 = \frac{\partial \Phi}{\partial r}, \quad \phi_1 = \frac{\partial \phi}{\partial r}, \quad \Delta\Phi = \Delta\phi = 0. \tag{65}$$

Conversely, if $\partial\Phi/\partial r$ and $\partial\phi/\partial r$ satisfy equation $\Delta_1(\cdot) = 0$, then Φ and ϕ are harmonic, provided that all these functions vanish at infinity.

Next, from the system (52) it follows that $\vartheta = p/\mu$ and ω may also be expressed as the derivatives of some harmonic function $\tilde{P}$:

$$\frac{p}{\mu} = \frac{\partial \tilde{P}}{\partial z}, \quad \omega = \frac{\partial \tilde{P}}{\partial r}, \quad \Delta\tilde{P} = 0. \tag{66}$$

Then Equation (58) takes the form

$$\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\frac{\partial \tilde{P}}{\partial z} + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\frac{\partial \phi}{\partial r} + \frac{\partial \chi_0}{\partial z} = 0.$$

By virtue of equality $\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = \omega = \frac{\partial \tilde{P}}{\partial r}\}$ we obtain yet another condition on the functions $\tilde{P}$, ϕ, and χ_0:

$$\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\frac{\partial \tilde{P}}{\partial r} - \frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial \chi_0}{\partial r} = 0.$$

Let us multiply the last two equations by dr and dz correspondingly and, taking into account that ϕ is harmonic, add them:

$$\begin{aligned}\frac{\partial \chi_0}{\partial r}\,\mathrm{d}r + \frac{\partial \chi_0}{\partial z}\,\mathrm{d}z &= \left\{-\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\frac{\partial \tilde{P}}{\partial r} + \frac{\partial^2 \phi}{\partial r\,\partial z}\right\}\mathrm{d}r \\ &\quad + \left\{-\left(3 + r\frac{\partial}{\partial r} + z\frac{\partial}{\partial z}\right)\frac{\partial \tilde{P}}{\partial z} + \frac{\partial^2 \phi}{\partial z^2}\right\}\mathrm{d}z.\end{aligned}$$

The obtained equality is nothing but the complete differential of function χ_0, which allows us to express χ_0 explicitly in terms of functions $\tilde{P}$ and ϕ:

$$\chi_0 = -2\tilde{P} - r\frac{\partial \tilde{P}}{\partial r} - z\frac{\partial \tilde{P}}{\partial z} + \frac{\partial \phi}{\partial z} + \text{ const.} \tag{67}$$

To ensure the proper behavior of the functions (67) at infinity, the constant of integration in (67) must be set to zero: const $= 0$. On inserting the expression (67) for function χ_0 in (57), the general solution (57)–(58) takes the form

$$u_r = \frac{1}{2}\Big(r\frac{\partial \tilde{P}}{\partial z} + \frac{\partial \phi}{\partial r} \Big), \quad u_z = \frac{1}{2}\Big(-2\tilde{P} - r\frac{\partial \tilde{P}}{\partial r} + \frac{\partial \phi}{\partial z} \Big),$$

which by substitution

$$\tilde{P} = -2P, \quad \phi = -2\frac{\partial \Phi}{\partial z}, \tag{68}$$

reduces to the general solution (44), (47) with function Ψ. From (65) and the second equality of (68) it follows that the original functions Φ_1 and ϕ_1 satisfy

$$\phi_1 = -2\frac{\partial \Phi_1}{\partial z}.$$

Differentiating function Φ_1 with respect to z yields the relationship between the coefficients of ϕ_1 and Φ_1:

$$\tilde{B}_{n,0} = \frac{1}{c}\left\{(n+3/2)\varpi_{n+1} - 2n\varpi_n + (n-3/2)\varpi_{n-1}\right\}, \quad n \geq 1.$$

Replacing $\tilde{B}_{n,0}$ in the last equality with $\tilde{x}_{n,0}$ similarly to (19), and expressing ϖ_n by g_n and h_n, using Equations (50), we obtain the exact solution of system (61) in an explicit form:

$$\tilde{x}_{n,0} = -\frac{1}{c\,\lambda_{n,0}}\left\{(n-3/2)(g_n+h_n) + 3\sum_{j=0}^{n}(g_j+h_j)\right\}, \quad n \geq 1. \tag{69}$$

Note that the form of the solution (69) closely resembles the formula (28) (recall that the finite sum in (69) can be transformed to the infinite sum using relationships (54) and (55)).

7. Conclusions

We have presented an exact analytical solution of the displacement boundary-value problem of elasticity for a torus. The original boundary-value problem was ultimately reduced to an infinite system of algebraic equations with tridiagonal matrices, which are the simplest equations that can arise in a vector boundary-value problem of elasticity in toroidal regions. The last proposition rests on two observations: (i) a harmonic boundary-value problem in a toroidal region function cannot be reduced to an infinite system with less than three diagonals, unless the value of the sought function is prescribed at the whole boundary, and (ii) the displacement boundary-value problem of elasticity cannot be reduced to pure Dirichlet problems for harmonic functions. Therefore, the obtained tridiagonal infinite systems can be considered to represent an exact solution of the original boundary-value problem in terms of the solvability of basic problems of the potential theory in toroidal regions. Moreover, we have presented two analytical techniques for solving tridiagonal systems, with one technique being potentially applicable to systems with a greater number of diagonals.

Reduction of the original displacement boundary-value problem for a torus to equations of the simplest form was possible owing to the special form of the derived general solution of the elastostatics equation. This general solution presents the vector of elastic displacement as a linear combination of a vector and scalar harmonic functions. However, these functions are not independent and have to satisfy an additional differential constraint. The presence of the differential constraint to be satisfied by the harmonic functions makes the introduced general solution different from other types of general solution of the elastostatics equation. Although we have concentrated only on the problem for a torus, the developed general approach is applicable to problems for other bodies described by cyclidal coordinates (lens-shaped body, spindle-shaped body, and bi-spheres) and will produce similar results.

However, it has been shown that the double-connectedness of the torus may lead to non-unique solutions of certain vector boundary-value problems, such as elasticity problems that involve the generalized Cauchy-Riemann equations. In Section 5 we have determined conditions that ensure single-valuedness of the solutions of the generalized Cauchy-Riemann equations, and have shown that these conditions are satisfied identically for the introduced form of the general solution.

In the considered example we have presented a complete solution of the Stokes problem for a torus, using both the traditional stream-function approach and the developed elastic solution. It has been shown that, in contrast to the presented approach, the stream-function approach does not yield a single-valued solution, and requires the obtained uniqueness conditions for the generalized Cauchy-Riemann equations to be satisfied. Finally, we have demonstrated a direct relation between both types of general solution for the Stokes equations, and used this relation to derive an explicit solution for the tridiagonal system obtained by application of the elastic solution to the Stokes problem for a torus.

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Appendix. Exact solution of the boundary-value problem for function Ψ

Coefficients $g_0^{(1,2)}$ and $h_0^{(1,2)}$ of (51) have the following form:

$$g_0^{(1)} = \frac{2B^*}{\Upsilon_0}\left\{2P_{\frac{1}{2}}\left(Q^1_{\frac{1}{2}} - \cosh\xi_0\, Q^1_{-\frac{1}{2}}\right) - \cosh\xi_0\left(3P_{-\frac{1}{2}}\, Q^1_{\frac{1}{2}} + P^1_{-\frac{1}{2}}\, Q_{\frac{1}{2}}\right) - 3\frac{\cosh^2\xi_0}{\sinh\xi_0}\right\},$$

$$g_0^{(2)} = \frac{2B^*}{\Upsilon_0}\left\{-\frac{2}{3}P_{\frac{1}{2}}\left(Q^1_{\frac{1}{2}} + 3\cosh\xi_0\, Q^1_{-\frac{1}{2}}\right) + \cosh\xi_0\left(P_{-\frac{1}{2}}\, Q^1_{\frac{1}{2}} + 3P^1_{-\frac{1}{2}}\, Q_{\frac{1}{2}}\right) - 3\frac{\cosh^2\xi_0}{\sinh\xi_0}\right\},$$

$$h_0^{(1)} = \frac{2B^*}{\Upsilon_0}\left\{\frac{1}{\sinh\xi_0} - 2P_{\frac{1}{2}}\, Q^1_{\frac{1}{2}} + 3\cosh\xi_0\left(P^1_{\frac{1}{2}}\, Q_{-\frac{1}{2}} + P_{\frac{1}{2}}\, Q^1_{-\frac{1}{2}}\right)\right\},$$

$$h_0^{(2)} = \frac{2B^*}{\Upsilon_0}\left\{-\frac{1}{\sinh\xi_0} - 2P^1_{\frac{1}{2}}\, Q_{\frac{1}{2}} + 3\cosh\xi_0\left(P^1_{\frac{1}{2}}\, Q_{-\frac{1}{2}} + P_{\frac{1}{2}}\, Q^1_{-\frac{1}{2}}\right)\right\},$$

where $B^* = \dfrac{\sqrt{2}}{\pi}cV$ and

$$\Upsilon_0 = 2P^1_{\frac{1}{2}}\, P_{\frac{1}{2}} - 3\cosh\xi_0\left(P^1_{\frac{1}{2}}\, P_{-\frac{1}{2}} + P_{\frac{1}{2}}\, P^1_{-\frac{1}{2}}\right).$$

For $n \geq 1$ the expressions for $g_n^{(1,2)}$ and $h_n^{(1,2)}$ are

$$g_n^{(1)} = \frac{B^*}{n\Upsilon_n}\left\{-\frac{(n+\frac{1}{2})(n+\frac{3}{2})}{\sinh\xi_0} + G_n\right\}, \quad g_n^{(1)} = \frac{B^*}{n\Upsilon_n}\left\{-\frac{(n-\frac{1}{2})(n-\frac{1}{2})}{\sinh\xi_0} + H_n\right\},$$

$$g_n^{(2)} = \frac{4B^*}{3n(n-\frac{1}{2})(n-\frac{3}{2})\Upsilon_n}\left\{-\frac{(n-\frac{1}{2})(n-\frac{3}{2})}{\sinh\xi_0} + G_n\right\},$$

$$h_n^{(2)} = \frac{4B^*}{3n(n+\frac{1}{2})(n+\frac{3}{2})\Upsilon_n}\left\{-\frac{(n+\frac{1}{2})(n+\frac{3}{2})}{\sinh\xi_0} + H_n\right\},$$

where

$$\begin{aligned}
G_n &= (n-\tfrac{1}{2})(n-\tfrac{3}{2})P^1_{n+\frac{1}{2}}\,Q_{n-\frac{3}{2}} - (n+\tfrac{1}{2})(n+\tfrac{3}{2})P_{n+\frac{1}{2}}\,Q^1_{n-\frac{3}{2}},\\
H_n &= (n+\tfrac{1}{2})(n+\tfrac{3}{2})P^1_{n-\frac{3}{2}}\,Q_{n+\frac{1}{2}} - (n-\tfrac{1}{2})(n-\tfrac{3}{2})P_{n-\frac{3}{2}}\,Q^1_{n+\frac{1}{2}},\\
\Upsilon_n &= (n+\tfrac{1}{2})(n+\tfrac{3}{2})P_{n+\frac{1}{2}}\,P^1_{n-\frac{3}{2}} - (n-\tfrac{1}{2})(n-\tfrac{3}{2})P^1_{n+\frac{1}{2}}\,P_{n-\frac{3}{2}}.
\end{aligned}$$

In the above formulas all Legendre functions are of argument $\cosh\xi_0$.

References

1. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics.* New York: McGraw-Hill (1953) 997 pp.
2. A. Wangerin, Über das Problem des Gleichgewichts elastischer Rotationskörper. *Arch. der Math. und Phys.* 55 (1873) 113–146.
3. Yu. I. Soloviev, The axisymmetric problem of elasticity for a torus and a space with toroidal cave. *Mekh. Tverd. Tela* 3 (1969) 99–105 [in Russian].
4. Yu. N. Podil'chuk and V. S. Kirilyuk, Nonaxially symmetric deformation of a torus. *Soviet Appl. Mech.* 19 (1983) 743–748.
5. V. S. Kirilyuk, Stress concentration in an isotropic medium with an elastic toroidal inhomogeneity. *Soviet Appl. Mech.* 24 (1988) 11–14.
6. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics.* New York: Prentice Hall (1965) 556 pp.
7. W. H. Pell and L. E. Payne, On Stokes flow about a torus. *Mathematika* 7 (1960) 78–92.
8. S. J. Wakiya, On the exact solution of the Stokes equations for a torus. *J. Phys. Soc. Jap.* 37 (1974) 780–783.
9. N. Lebedev, The functions associated with a ring of oval cross-section. *Tech. Phys.* 1 (1937) 3–24.
10. E. W. Hobson, *Spherical and Ellipsoidal Harmonics.* Cambridge: University Press (1931) 500 pp.
11. H. Bateman and A. Erdelyi. *Higher Transcendental Funcitons. Vol. 1.* New York: McGraw-Hill (1953) 282 pp.
12. G. V. Kutsenko and A. F. Ulitko, An Exact Solution of the Axisymmetric problem of the theory of elasticity for a hollow ellipsoid of revolution. *Soviet Appl. Mech.* 11 (1975) 3–8.
13. F. D. Gakhov, *Boundary Value Problems.* New York: Pergamon Press (1966) 564 pp.
14. N. I. Muskhelishvili, *Singular Integral Equations.* Groningen: Noordhoff (1953) 447 pp.
15. G. G. Stokes, *Mathematical and Physical Papers. Vol. 1.* Cambridge: University Press (1880) 352 pp.
16. Ghosh, S. On the steady motion of a viscous liquid due to translation of a tore parallel to its axis. *Bull. of the Calcutta Math. Soc.* 18 (1927) 185–194.
17. H. Takagi, Slow viscous flow due to the motion of a closed torus. *J. Phys. Soc. Jap.* 35 (1973) 1225–1227.